Final Review

1. (10 points) Find the general solution of

$$\begin{aligned} x + 2y + z &= 1 \\ -x - 2y + z &= 2 \\ 2x + 4y + 2z &= 2 \end{aligned}$$

2. Find the inverse and the determinant of $A = \begin{bmatrix} -6 & 2 & -2 \\ 2 & -2 & 0 \\ -2 & 0 & 2 \end{bmatrix}$

3. Suppose A is a 3×3 invertible matrix and $A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 5 & 2 \\ 1 & 0 & 3 \end{bmatrix}$.

   (a) Solve the system of equations $AX = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

   (b) Solve the system of equations $A^T X = \begin{bmatrix} 2008 \\ 1 \\ 0 \end{bmatrix}$

4. Suppose A and B are 4x4 matrices. If $det(A) = -2$ and $det(B) = 3$ find

   (a) $det(3A^{-1}.B^{-1})$
   (b) $det(A^2.4B^{-1})$
   (c) $det(adj(A))$

5. Suppose A is a 2×2 invertible matrix. If the row operation $-2R_1 + R_2 \to R_2$ and then the row operation $-R_2 + R_1 \to R_1$ is performed on A, it becomes the identity matrix.

   (a) Find two elementary matrices E_1 and E_2 such that $E_2 E_1 A = I_{2\times 2}$.
   (b) Write A as a product of elementary matrices. [Hint: Use (a)]
   (c) Write A^{-1} as a product of elementary matrices. [Hint: Use (a)]

6. (10 points) For which values of x (if any) is the matrix $\begin{pmatrix} 1 & 0 & -3 \\ 0 & x & 2 \\ 3 & -10 & x \end{pmatrix}$ singular (not invertible)?

7. (12 points) Express

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 4 & 0 \\ 3 & 0 & 4 \end{pmatrix}$$

as a product of elementary matrices.

8. Find a basis for the subspace of R^4 spanned by

$$\{(2,9,-2,53),\ (-3,2,3,-2),\ (8,-3,-8,17),(0,-3,0,15)\}$$

9. Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 2 & 1 \\ 2 & -6 & 4 \end{bmatrix}$, find

   (a) the rank of the matrix

   (b) a basis for Nul(A).

   (c) a basis of the row space of A.

   (d) a basis for the column space of A.

10. Suppose $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 5 & 2 & 3 \\ 1 & 0 & 3 & 2 \end{bmatrix}$ and echelon form of A is $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

    (a) Find a basis for the column space of A.

    (b) Find the nullity of A.

    (c) Find a basis for the row space of A.

11. (12 points) Which of the following are subspaces of the given vector space V? Justify your answers.

    (a) $V = R^3$, $S = \{(x,y,0) : x+y=0\}$.

    (b) $V = R^3$, $S = \{(x,y,0) : xy \geq 0\}$.

    (c) $V = R^{2\times 2}$, $S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : \text{All matrices in } V \text{ where } abc = 0 \right\}$

12. (10 points) Find a Basis for each of the following subspaces S of the given vector space V.

    (a) $V = R^{2\times 2}$, $S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : \text{All matrices in } V \text{ where } a+b-c=0 \right\}$

    (b) $V = P_2$, $S = \{\ p(x)$ is in P_2 and $p(1) = 0\}$

13. (12 points)Determine whether the following sets of vectors are linearly independent in the vector space V. Justify your answers.

    (a) $V = R^3$, $v_1 = (1,0,0),\ v_2 = (1,1,1)\ , v_3 = (2,2,3)$.

    (b) $V = P_3$, $p_1(x) = x^2 + x - 1, p_2(x) = 1 - x^2, p_3(x) = x$

14. (10 points) Let T be the linear transformation from R^2 to R^2 given by $T(a, b) = (a - b, 2b + a)$.

    (a) Find the standard matrix representation of T.

    (b) Find $T^{-1}(x, y)$ if T^{-1} exists.

15. (14 points) Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

    (a) Find the characteristic Polynomial of A.

    (b) Hence find the Eigenvalues of A.

    (c) For each Eigenvalue of A, find a basis of the corresponding Eigenspace.

    (d) Decide if A is diagonalizable or not. Justify your answer. If yes, give an invertible matrix P and a diagonal matrix D such that $PDP^{-1} = A$.

16. (10 points) Consider the vectors $u = (a - 1\ ,\ 1\ ,\ b)$, $v = (2\ ,\ a\ ,\ -1)$ and $w = (3\ ,\ a + b\ ,\ 2)$ in R^3. Find all values of a and b that make u orthogonal to both v and w.

17. Let $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, and $x = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$.

    (a) Show that the set $\beta = \{u_1, u_2, u_3\}$ is an orthogonal basis for R^3.

    (b) Express x as a linear combination of the elements in β.

18. Suppose $\beta = \left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -5 \\ 9 \\ -9 \\ 3 \end{bmatrix} \right\}$ is a basis for a subspace W. Use Gram-Schmidt to construct an orthonormal basis for W.